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Lecture Notes

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Different types of differential equations in respect of our atmospheric prediction equations.: We all know that an equation which involves derivative or differential of the dependent variable is called a differential equation. If the dependent variable is a function of only one independent variable, then we don't have any scope to talk about derivative or differentials with respect to multiple independent variables, in other words, we can talk of derivatives or differentials with respect to single variable only. Differential equations which involves derivative or differential of the dependent variable with respect to a single variable is called an ordinary differential equation (ODE). Examples of such equations are given below:

$$A\left(\frac{d^2u}{dt^2}\right)^n + B$$

Partial differential equation (PDE):

A PDE is an equation which involves partial derivatives or differentials of the dependent variable.

EX: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$ is a partial differential equation, as it contains the partial derivatives of the dependent variables u, p.

Order of a PDE :

It is the highest order partial derivatives involved in the equation.

Ex. Consider the PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x, y)$.

Here, u is the dependent variable, x, y are independent variables and F(x ,y) is a known function of x,y. In the PDE the highest order partial derivative involved in this equation is 2. So the order of this PDE is 2.

Linear and non-linear PDE :

A general form of a 2nd order PDE is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G.$$

In the above equation A,B,C,D,E,F and G are called coefficients of the PDE. If all these coefficients are constants or functions of independent variables (x , y), then the resulting PDE is known as a Linear PDE.

For example, let us consider the following PDE:

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

For this PDE A = 1, B = 2, C = 1 and D = E = F = G=0. Hence this PDE is a Linear PDE.

We consider another PDE,

$y^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} = (x+y)$. In this PDE, A, B, C and G are functions of x or y or both. So, this is also a 2nd order non-linear PDE.

On the other hand, if at least one these coefficients is a function dependent variable, then the resulting PDE is known as a non-linear PDE.

For example, let us consider the following PDE:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} .$$

In the above equation, A = B = C = F = 0, D = u and E = v. Since u, v are dependent variables, hence it is a non-linear PDE. Thus, governing equations are non-linear partial differential equation.

Above nonlinear PDEs can't be solved analytically, one of the reasons of which is that the analytical expressions of the coefficients are unknown. Hence, these model equations are integrated forward in time using numerical method and spectral method.

a. In numerical method first the continuous time and 3-D space domain are discretized, like, $\{(x, y, z): (x, y, z) \in R^3\} \rightarrow \{(i\Delta x, j\Delta y, k\Delta z): (i, j, k) \in \mathbb{Z}^3 \& \Delta x, \Delta y, \Delta z \ given \}$ and $\{t: 0 \ll t < \infty\} \rightarrow \{n\Delta t: n \in \mathbb{Z} \ \& \Delta t \ given \}$. The discrete spatial points $(i\Delta x, j\Delta y, k\Delta z)$ are denoted by (i, j, k) and called (i, j, k) grid point. Similarly, the discrete time $n\Delta t$ is called 'n' th time step. In numerical method values of the field variables (u, v, w, p, T, q, ρ) are specified at all discrete grid points at the time step '0' (initial time). Using these values of the field variables at different grid points at a given time step, spatial derivatives of the field variables are approximated numerically using a suitable finite difference scheme (FDS), for specifying the right-hand sides of the equations completely. This is followed by numerical integration in time for predicting values of the variable valid at next time step.

b. Different finite difference scheme:

i. Forward differencing scheme (FDS):

$$\left(\frac{\partial f}{\partial t}\right)^n_{(i,j,k)} \approx \frac{f_{ijk}{}^{n+1} - f_{ijk}{}^n}{\Delta t}, \left(\frac{\partial f}{\partial x}\right)^n_{(i,j,k)} \approx \frac{f_{(i+1)jk}{}^n - f_{ijk}{}^n}{\Delta x} \text{ etc.}$$

Error$\sim O(\Delta x, \Delta y, \Delta z, \Delta t)$

ii. Backward differencing scheme (BDS):

$$\left(\frac{\partial f}{\partial t}\right)^n_{(i,j,k)} \approx \frac{f_{ijk}{}^{n} - f_{ijk}{}^{n-1}}{\Delta t}, \left(\frac{\partial f}{\partial x}\right)^n_{(i,j,k)} \approx \frac{f_{ijk}{}^n - f_{(i-1)jk}{}^n}{\Delta x} \text{ etc.}$$

Error$\sim O(\Delta x, \Delta y, \Delta z, \Delta t)$

iii. Central differencing scheme or leap frog scheme (LFS):

$$\left(\frac{\partial f}{\partial t}\right)^n_{(i,j,k)} \approx \frac{f_{ijk}{}^{n+1} - f_{ijk}{}^{n-1}}{2\Delta t}, \left(\frac{\partial f}{\partial x}\right)^n_{(i,j,k)} \approx \frac{f_{(i+1)jk}{}^n - f_{(i-1)jk}{}^n}{2\Delta x} \text{ etc.}$$

Error$\sim O[(\Delta x)^2, (\Delta y)^2, (\Delta z)^2, (\Delta t)^2]$

Thus, LFS converges at a faster rate than that by FDS or BDS.

- **A few important concepts about FDS:**

  Consistency or compatibility of a FDS: If the FD approximation of derivative tends to its exact value or analytical value at each point / at each time as $\Delta x, \Delta y, \Delta z, \Delta t \to 0$

Convergence: Numerical solution of a well posed IVP is said to be convergence if it tends to analytical or exact solution as $\Delta x, \Delta y, \Delta z, \Delta t \to 0$

Lax equivalence theorem: Given a well posed IVP and a consistent FDS; then numerical solution is convergent if and only if it is stable, i.e., as number of time step $(n) \to \infty$, at each point.

Explicit & implicit time differencing scheme: To understand the concept of implicitness or explicitness of a differencing scheme, we refer the linear advection equation, viz.,

$\frac{\partial f}{\partial t} = -c\frac{\partial f}{\partial x}$. With c as constant phase speed.

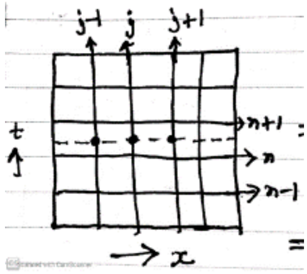
If the above equation is numerically approximated at the grid point $i\Delta x$ and at time step $n\Delta t$ using using following two FDS, it is seen that.

LFS: $\frac{u_i^{n+1}-u_i^{n-1}}{2\Delta t} = -c\frac{u_{i+1}^{n}-u_{i-1}^{n}}{2\Delta x} \Rightarrow u^{n+1} = f(u^n, u^{n-1})$

FDS: $\frac{u_i^{n+1}-u_i^{n}}{\Delta t} = -c\frac{u_{i+1}^{n}-u_{i}^{n}}{\Delta x} \Rightarrow u^{n+1} = f(u^n)$

Thus, using above schemes, future time step value can be found out by present and past time, using marching method. Such scheme, where future time step value can be found out by present and past time, is known as explicit scheme.

Now the time derivative of linear advection equation is approximated numerically using forward difference scheme, whereas space derivative is approximated using central difference scheme averaged between time steps 'n' & '(n+1)', as follows:



$$\frac{u_i^{n+1}-u_i^{n}}{\Delta t} = -c\left[\frac{\frac{\left(u_{i+1}^{n+1}+u_{i+1}^{n}\right)}{2}-\frac{\left(u_{i-1}^{n+1}+u_{i-1}^{n}\right)}{2}}{2\Delta x}\right]$$

$$\Rightarrow u^{n+1} = f(u^n, u^{n+1})$$

Thus, value of the variable at a grid point at future time step (n+1)

Requires present value of the variable at the grid point and future value at neighbouring grid points. Such scheme is known as implicit scheme.

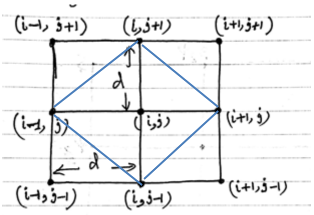
Numerical approximation of Laplacian: Laplacian of a scalar field $f(x, y)$ at any point (x,y) is given by, $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. Its numerical approximate value at an arbitrary grid point (i, j) is given by: $(\nabla^2 f)_{i,j} = \frac{f_{(i+1,j)}+f_{(i-1,j)}+f_{(i,j+1)}+f_{(i,j-1)}-4f_{(i,j)}}{d^2}$; where 'd' is the grid length.

- ❖ Numerical approximation of Jacobean: Let us consider two scalar fields, ψ & S. Jacobean of these two fields, denoted by $J(\psi, S)$ is given by

$$J(\psi,S)=\frac{\partial\psi}{\partial x}\frac{\partial S}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial S}{\partial x}\ldots(1)$$

$$=\frac{\partial}{\partial x}\left(\psi\frac{\partial S}{\partial y}\right)-\frac{\partial}{\partial y}\left(\psi\frac{\partial S}{\partial x}\right)\ldots(2)$$

$$=\frac{\partial}{\partial y}\left(\mathrm{S}\frac{\partial\Psi}{\partial x}\right)-\frac{\partial}{\partial x}\left(\mathrm{S}\frac{\partial\Psi}{\partial y}\right)\ldots(3)$$



Numerical approximate value of the expression (1), (2) & (3) of the Jacobean, using above figure, are given below:

$$J_1=\left[\frac{(\psi_{(i+1,j)}-\psi_{(i-1,j)})(S_{(i,j+1)}-S_{(i,j-1)})-(S_{(i+1,j)}-S_{(i-1,j)})(\psi_{(i,j+1)}-\psi_{(i,j-1)})}{4d^2}\right]$$

$$J_2=\left[\frac{\{\psi_{(i+1,j)}(S_{(i+1,j+1)}-S_{(i+1,j-1)})-\psi_{(i-1,j)}(S_{(i-1,j+1)}-S_{(i-1,j-1)})\}-\{\psi_{(i,j+1)}(S_{(i+1,j+1)}-S_{(i-1,j+1)})-\psi_{(i,j-1)}(S_{(i+1,j-1)}-S_{(i-1,j-1)})\}}{4d^2}\right]$$

$$J_3=\left[\frac{\{S_{(i,j+1)}(\psi_{(i+1,j+1)}-\psi_{(i-1,j+1)})-S_{(i,j-1)}(\psi_{(i+1,j-1)}-\psi_{(i-1,j-1)})\}-\{S_{(i+1,j)}(\psi_{(i+1,j+1)}-\psi_{(i+1,j-1)})-S_{(i-1,j)}(\psi_{(i-1,j+1)}-\psi_{(i-1,j-1)})\}}{4d^2}\right]$$

Arakawa Jacobean at the (I,j) grid point is given as the average of J_1, J_2 & J_3.

Linear Computational stability analysis: For linear computational instability analysis, we refer the following linear advection equation $\frac{\partial f}{\partial t}=-c\frac{\partial f}{\partial x}$, given initial condition $f(x,o)=Ae^{i\mu x}$ and c is constant phase speed. When this equation is solved analytically using the method of separation of variables, we get the analytical solution, $f(x,t)=Ae^{i\mu(x-ct)}$. it is obvious that this analytical solution is absolutely stable. Now, when the above equation is attempted to solve numerically using leap frog scheme, then it can be shown that the numerical solution is stable if $\frac{c\Delta t}{\Delta x}<1$. Even if it is attempted using other explicit scheme then also similar type of condition for stability can be obtained. Thus, numerical solution of above equation is conditional, when explicit difference scheme is used. However, when it is attempted to solve numerically using trapezoidal semi-implicit scheme as shown below:

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=-\mathrm{c}\left[\frac{\frac{(u_{j+1}^{n+1}+u_{j+1}^n)}{2}-\frac{(u_{j-1}^{n+1}+u_{j-1}^n)}{2}}{2\Delta x}\right].$$

Let, $u_j^n=B^{n\Delta t}e^{i\mu j\Delta x}$

$$(B^{\Delta t}-1)=-ic\frac{\Delta t}{2\Delta x}\sin(\mu\Delta x)\,[(B^{\Delta t}+1)].$$

Let, $c\frac{\Delta t}{\Delta x} = \sigma$

$$\frac{(B^{\Delta t} - 1)}{(B^{\Delta t} + 1)} = -\frac{i\sigma \sin(\mu\Delta x)}{2}$$

$B^{\Delta t} = \frac{2 - i\sigma\sin(\mu\Delta x)}{2 + i\sigma\sin(\mu\Delta x)} = \frac{4+\sigma^2\sin^2(\mu\Delta x) - 4i\sigma\sin(\mu\Delta x)}{4+\sigma^2\sin^2(\mu\Delta x)} \Rightarrow \left|B^{\Delta t}\right|^2 = 1 \Rightarrow \left|B^{\Delta t}\right| = 1.$

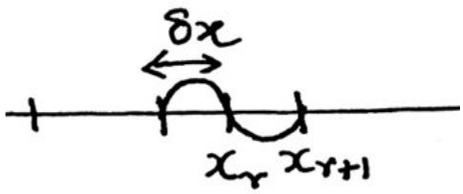
Thus, $\left|B^{\Delta t}\right|^n = 1\, for\ any\ time\ step\ 'n'$. Hence this scheme is unconditionally or absolutely stable.

Non-linear computational instability: Consider nonlinear advection equation

$\frac{\partial f}{\partial t} = -\mathrm{u}\frac{\partial f}{\partial x}$, where, u is a function of x,t. Let us consider a limited interval$[a, b]$ and be divided into 'N' equal segments, by inserting grid points,$a = x_0, x_1, x_2, \ldots\ldots, x_{n-1}, x_n = b$, with width δx between two arbitrary consecutive points. Then the wave length of shortest possible wave is $2\delta x$, as shown in adjoining figure.

Let the dependent variables be expressed as $u(x,t) = \sum_{k=1}^{n} a_{uk}\cos kx + \sum_{k=1}^{n-1} b_{uk}\sin kx$ and $f(x,t) = \sum_{k=1}^{n} a_{fk}\cos kx + \sum_{k=1}^{n-1} b_{fk}\sin kx$.

Then the product term $\mathrm{u}\frac{\partial f}{\partial x}$ will have terms like $sin(m+l)\,x, cos(m+l)\,x\ etc$. For some terms, $(m+l) > \frac{N}{2}$.



Such terms correspond to wave with wave length$< 2\delta x$. But the shortest wave, that can be represented with given grid arrangement is $2\delta x$. Thus a wave with wave length shorter than $2\delta x$ will be falsely represented by a relatively longer wave of wave length$2\delta x$.

This false representation of a shorter wave by a longer wave is known as aliasing. Repeated aliasing gives rise to nonlinear instability. It is due to the presence of nonlinear advection term $u\frac{\partial f}{\partial x}$.

Advection of a variable S can be expressed as $J(\psi, S)$.

$$\mathrm{J}(\psi, \mathrm{S}) = \frac{\partial\psi}{\partial x}\frac{\partial S}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial S}{\partial y} \ldots (1)$$

$= \frac{\partial}{\partial x}\left(\psi\frac{\partial S}{\partial y}\right) - \frac{\partial}{\partial y}\left(\psi\frac{\partial S}{\partial x}\right) \ldots (2)$

$=\frac{\partial}{\partial y}\left(S\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(S\frac{\partial\psi}{\partial y}\right) \ldots (3)$. These 3 expressions of J are approximated at (i, j)th grid point, numerically by say, J_1, J_2 &J_3. Arakawa Jacobian is defined by $J = \frac{J_1+J_2+J_3}{3}$. If the advection term is numerically approximated by Arakawa Jacobian, then this Aliasing and non-linear instability can be eliminated.

Numerical method of solving Poison's equation:

Poison's equation is stated as below:

Solve the equation, $\nabla^2 f = h(x,y)$ for the unknown function $f(x,y)$, where $h(x,y)$ is a known function.

Such type of equation is solved numerically using relaxation method, which is described below:

Numerically approximate form of the above equation at a grid point (i,j) is

$$\frac{f_{(i+1,j)} + f_{(i-1,j)} + f_{(i,j+1)} + f_{(i,j-1)} - 4f_{(i,j)}}{d^2} = h_{(i,j)}$$

This method starts with some initial guess values of the unknown function f(x,y) at all grid points. If, $f_{(i,j)}^{(0)}$ is the initial guess value of f(x,y) at any arbitrary grid point (i,j); then error in the initial guess, when substituted in the above equation, is given by

$$R_{(i,j)}^{(0)} = \frac{f_{(i+1,j)}^{(0)} + f_{(i-1,j)}^{(0)} + f_{(i,j+1)}^{(0)} + f_{(i,j-1)}^{(0)} - 4f_{(i,j)}^{(0)}}{d^2} - h_{(i,j)}$$

Above relation gives an improved guess value of f(x,y) at a grid point (i,j)

$$f_{(i,j)}^{(1)} = \frac{d^2}{4} R_{(i,j)}^{(0)} + f_{(i,j)}^{(0)}$$

Then, following above, the error in the first improved guess is given by

$$R_{(i,j)}^{(1)} = \frac{f_{(i+1,j)}^{(1)} + f_{(i-1,j)}^{(1)} + f_{(i,j+1)}^{(1)} + f_{(i,j-1)}^{(1)} - 4f_{(i,j)}^{(1)}}{d^2} - h_{(i,j)}$$

And subsequently the second improved guess value is obtained as

$$f_{(i,j)}^{(2)} = \frac{d^2}{4} R_{(i,j)}^{(1)} + f_{(i,j)}^{(1)}$$

The above iteration process is said to converges when two successive improved guess of the unknown function f(x,y) differs by a number smaller than a very small pre-assigned positive number, say, ε,i.e.,

$\left| f_{(i,j)}^{(m+1)} - f_{(i,j)}^{(m)} \right| < \varepsilon$, at every grid point (i,j). Then either of these two successive improved guess value may be treated as approximate numerical solution of Poison's equation at a grid point (i,j).

Using this method, knowing horizontal wind components (u,v) at different grid point, one can find out stream function (ψ), velocity potential (χ), rotational wind $(\overrightarrow{V_\psi})$ and divergent wind $(\overrightarrow{V_\chi})$, using following steps:

i. Compute vorticity at every grid point using horizontal wind components at each grid point

Vorticity(ς): $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \approx \left[\frac{v^n_{(i+1)jk} - v^n_{(i-1)jk}}{2\Delta x}\right] - \left[\frac{u^n_{i(j+1)k} - u^n_{i(j-1)k}}{2\Delta y}\right]$

Divergence $(D_h) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \approx \left[\frac{u^n_{(i+1)jk} - v^n_{(i-1)jk}}{2\Delta x}\right] + \left[\frac{v^n_{i(j+1)k} - v^n_{i(j-1)k}}{2\Delta y}\right]$

ii. Then set up the following poison's equations for the stream function (ψ) and velocity potential (χ) : $\nabla^2\psi = \zeta(x,y)$ and $\nabla^2\chi = -D_h(x,y)$. These two equations are solved to find out values of stream function (ψ) and velocity potential (χ) at each grid point.

iii.Then, rotational & divergent wind at any grid point are obtained as:

$V_\psi = \hat{\imath}\left(-\frac{\partial\psi}{\partial y}\right) + \hat{\jmath}\left(\frac{\partial\psi}{\partial x}\right) \approx \hat{\imath}\left(-\left[\frac{\psi^n_{i(j+1)k} - \psi^n_{i(j-1)k}}{2\Delta y}\right]\right) + \hat{\jmath}\left[\frac{\psi^n_{(i+1)jk} - \psi^n_{(i-1)jk}}{2\Delta x}\right]$ and

$$V_\chi = -\left[\hat{\imath}\left(\frac{\partial\chi}{\partial x}\right) + \hat{\jmath}\left(\frac{\partial\chi}{\partial y}\right)\right] \approx -\left\{\hat{\imath}\left[\frac{\chi^n_{(i+1)jk} - \chi^n_{(i-1)jk}}{2\Delta x}\right] + \hat{\jmath}\left[\frac{\chi^n_{i(j+1)k} - \chi^n_{i(j-1)k}}{2\Delta y}\right]\right\}$$